PRIVATE AND PUBLIC CONSUMPTION AND SAVINGS IN THE VON NEUMANN MODEL OF AN EXPANDING ECONOMY*

I. THE PROCESS OF EXPANSION: THE BASIC ISSUES

The von Neumann model of an expanding economy [12][1] has been recognized as one of the most fundamental contributions to mathematical economics. As a consequence it has produced a large literature. The purpose of the present paper is to extend further the economic interpretations and application of the model, based on work previously done by us and JOHN G. KEMENY [6] (subsequently quoted as KMT) in which a far reaching generalization of the original model was accomplished. While it would not be too difficult to make our new model, to be described in Section II, comprehensive enough to include all the models that followed, we do not choose to do so. In particular, we are not concerned with the so-called 'Turnpike Theorem' which is a special application of the von Neumann model. (For a summary of the literature and references see [3].) However, in Table 1, to be discussed below, we give an overview of the principal developments.

The purpose of studying models of the type under investigation is not at first to make them 'realistic', but rather to see whether such models are at all *logically possible* and what their properties would be. This is the correct procedure. Only after these properties have been described can one approach the question of proper realism. Sometimes a logically possible model may also yield immediately empirically relevant results.

The original von Neumann model settled the question whether uniform expansion is at all possible and proved the equality of expansion factor and interest factor. The Kemeny-Morgenstern-

Thompson generalization settled the same for a much less restrictive case than VON NEUMANN had analyzed. We have shown, in addition, how outside demand could be introduced, how its variation affects growth rates and interest rates; we allowed further for technological progress appearing in the form of new, more efficient processes, and described the possibilities and consequences of aggregation in the model. Perhaps most important, our model described the phenomenon of 'sub-economies', each one of which having its own expansion rate and interest rate, their total number proving to be finite. We have also shown the possibility of non-uniqueness of von Neumann balanced growth paths, of importance for the turnpike problems.

The method of analysis in KMT [6] was game theoretical, in the sense that game theory was used as an analytical, mathematical device of great power. The present model will use the same techniques and the same symbolism, but the presentation will be self-contained. We leave further interpretations of our new results to another occasion, believing that setting forth basic, elementary, modifications of the assumptions is indicated before entering upon more detailed interpretation. The main task is still to explore what is logically possible, in the spirit of the relevant remark made above.

The new model answers the question of the influence upon the expansion of varying private and public savings and consumption rates, conditions so far not investigated in any of the previous models listed in Table 1 below. It yields among other things the surprising result of alternative solutions such that the choice from among them is dependent not on any characteristics of the activities contained in the model but on outside circumstances and considerations, as will be described below. In this is seen the power of mathematical analysis since this result could not have been found by any other method of reasoning.

It is necessary to restate some fundamentals concerning different types of economies and their relation to the process of expansion: In a centrally directed economy decisions as to kinds and quantities of goods to be produced and their prices are made by a central planning committee. Therefore, indirectly—and sometimes even explicitly— the committee decides what the expansion rate of the economy shall be. But in a free exchange economy no such committee exists. Therefore the corresponding decisions are made, somehow, by the economy

itself. Classical and neo-classical economics has concerned itself with the problem of how consumers' decisions or preferences make themselves felt, via demand curves and thus affect prices and production. But nowhere in this mechanism[2] is there an explanation of how consumer preferences determine or even influence things such as the expansion rate or the interest rate in a multisector economy. There have been, of course, many attempts to explain *change*, e.g., via changes in tastes, in technology, in income distribution and there have been efforts to interpret change as expansion (e.g., by means of finding new markets, etc.). But there are no rigorous theorems. It is one of the purposes of the present paper to suggest an explanation of the phenomenon of consumers' influence via variable savings and consumption ratios.

Classical economics has also asserted the existence of sub-economies—that is, of self-sufficient segments of the economy that can operate as miniature economies by themselves. Examples are individual households, firms, regions of a country, etc. Again, in a centrally directed economy the central planning committee could decide that the economy should operate just one of its possible sub-economies if there were political, ideological or military reasons for doing so. In a completely free exchange economy there is no way of taking any corresponding action. In a mixed economy, where there is a considerable amount of government intervention, certain price and quantity decisions can be enforced, particularly in times of stress. Nevertheless, even in a free exchange economy we can observe that a 'decision' has been made what sub-economy of the potential economy shall be used. It is once more the consumers who affect these decisions via their consumption decisions. In addition, the government provides public goods according to public demand. Classical economics again fails to provide an explanation of the mechanism by which this is effected. What attempts there were have been based on production functions, not on ultimate consumers' preferences

2. The word 'mechanism' should be used *cum grano salis*; i.e., there is no intended parallelism with physical mechanics. The word is used here for want of a commonly accepted, more appropriate term. Game theory has shown that the structure of social theory is quite different from those classical ideas normally associated with mechanics. Consequently, adjustment processes also obey different rules and have different appearances.

although the latter are of primary importance. However, it would not be too difficult to show how such a mechanism might work, in conformity with the results about sub-economies obtained in KMT.

It is useful at this point to consider the following. In a free exchange economy the preferences of the consumers for private and public goods guide the entire course of the economy. If these preferences, together with technology and the supply of necessary inputs, warrant the economy will expand—presumably at the maximum rate consistent with the above preferences.

If the government, or some members of the economy desire a still higher expansion rate, the preferences must be changed, voluntarily or by interference. The government may set up an infeasible program. If a conflict exists between the consumers and the government we have a true game situation which has to be treated as such. This aspect has as yet not been introduced although it is of a very fundamental nature.
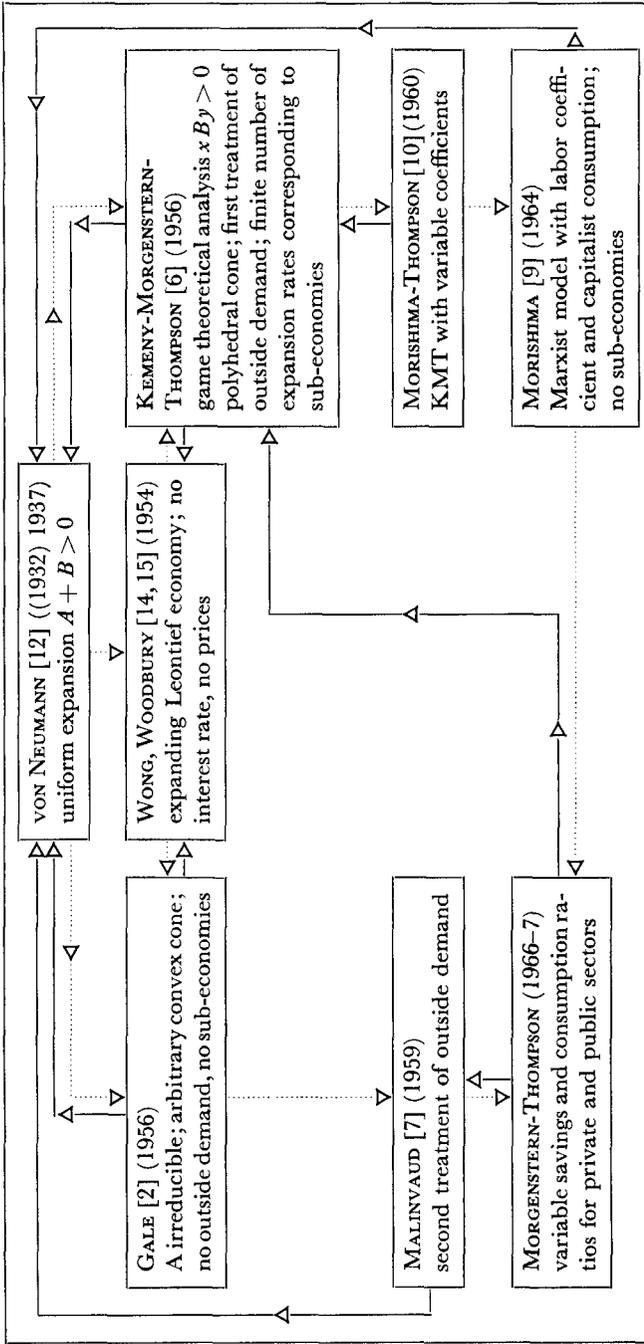
Let us emphasize that we regard our present contribution as neither conclusive nor complete. Rather, it is another step in the enrichment of interpretations of the von Neumann model that we hope will eventually lead to its complete assimilation into mathematical economics.

In order to show the interrelations of the original model to subsequent developments we list in Table 1 those works in which new theorems are proved relating directly to the original von Neumann model. Applications, as, for example, the more special turnpike theorem, are not considered here. The historical sequences are shown by dotted lines; the solid lines indicate, as the arrow points, how one model is contained in the other. One observes that the present model is the most general. It is capable of further explicit descriptions than those given here, e.g., regarding sub-economies; these will be given at another occasion.

Specifically, we make these further comments on the following authors:

1. GALE [2]. The principal differences from the KMT model are the following:
   (a) Gale permits an arbitrary convex cone of production possibilities while KMT assume a *polyhedral* convex cone.

Table 1

VON NEUMANN [12] ((1932) 1937) uniform expansion $A + B > 0$

KEMENY-MORGENSTERN-THOMPSON [6] (1956) game theoretical analysis $x\, By > 0$ polyhedral cone; first treatment of outside demand; finite number of expansion rates corresponding to sub-economies

MORISHIMA-THOMPSON [10] (1960) KMT with variable coefficients

MORISHIMA [9] (1964) Marxist model with labor coefficient and capitalist consumption; no sub-economies

WONG, WOODBURY [14, 15] (1954) expanding Leontief economy; no interest rate, no prices

GALE [2] (1956) A irreducible; arbitrary convex cone; no outside demand, no sub-economies

MALINVAUD [7] (1959) second treatment of outside demand

MORGENSTERN-THOMPSON (1966–7) variable savings and consumption ratios for private and public sectors

Note: All arrows transitive: $\to$ chronological; $\rightarrow$ implication.

The publications listed in this table only contain those works in which new theorems are proved relating directly to the original von Neumann model. Applications, as, for example, the more special turnpike theorem, are not considered here.

Figure 1 below gives another way of comparing these papers by locating them on the c, s (consumption and savings coefficients) plane. It is interesting to note that all models *prior* to the one described in the present paper are located on the perimeter of the square $0 \leq c, s \leq 1$. The model contained in the present paper applies to the entire square.

(b) Gale is interested in (i) the maximum alpha—the expansion factor—and (ii) the minimum beta—the interest factor. He shows that the minimum beta is less than or equal to the maximum alpha.

(c) Gale ignores the KMT conditions $xBy > 0$, (used again in the present paper, cf. II below) and instead imposes for some of his theorems the much stronger condition $y > 0$, i.e., regularity. The latter condition obviously implies (but is not implied by) $xBy > 0$. For regular models, of course, min beta = max alpha.

(d) He discusses the concept of submodels, but assumes that every submodel has all goods as outputs—a very strong condition.

2. MALINVAUD [7]. He replaces in the outside demand condition of KMT, $(B - \beta A)y \leq \alpha d$, the right-hand side by zero. This states that even with outside demand there will only be a profitless economy. He restricts himself to a polyhedral cone and ignores the condition $xBy > 0$. In other respects Malinvaud follows Gale. Later we will show how his model fits into the present model.

3. HOWE's proof [5, as rapporteur for a seminar group] is entirely equivalent to that given in [6] and [11]. The change is merely formal and involves only expository originality.

4. MORISHIMA [9] has an extreme Marxist model of an economy in which (a) every process has a positive labor input; (b) labor only consumes, never saves; (c) capitalists can consume if they make a profit, and if they choose to save, their savings are invested *outside* the economy. The influence of the Morishima-Thompson paper [10] is evident in this work. It is also interesting to note that the technique of proof used by Morishima, which utilizes the Eilenberg-Montgomery fixed point theorem, is not adequate for the case (which we treat in the present paper) in which investment of savings (either by worker or capitalist) is permitted within the economy. It is a technical point worth noting for economists who have found such fixed point theorems useful in other instances, that mappings derived from the present model need be neither lower semi-continuous nor contractible (c.f. Example 5 in Section V), and hence neither the von Neumann-Kakutani nor the Eilenberg-Montgomery theorem are applicable.

The literature on 'growth' is, of course, enormous and deals, for the most part qualitatively, with many other facets of that complex

phenomenon. Excellent surveys are found in [1] and [3]. As frequently the case in science, theoretical development requires, at first, strict limitation to simple, mathematically manageable cases. Therefore, we do not hesitate to restrict ourselves to the conditions set forth in the subsequent section, which though still few compared to full reality, exceed those previously made.

II. THE NEW MODEL: PRIVATE SECTOR

In the present section we shall develop a general model that includes, as special cases, both the Malinvaud model and KMT outside demand model.

As in the original KMT model, we shall let A and B be nonnegative $m \times n$ matrices whose entries are the input and output coefficients of the various industries respectively. We now introduce a nonnegative $m \times n$ matrix W whose entries will be interpreted as the excess profits of that industry. Thus, w_{ij} is the amount of good j obtained by workers in the i-th industry if that industry is run at unit intensity. The interpretation may be given as the amounts that stockholders in the firm purchase of each of the goods, using their dividends. It may be still further extended to include the amounts the firm reinvests in its own and other firms. The fact that w_{ij} depends upon both i and j reflects the fact that different industries may be located in different parts of the country and that regional tastes can prevail.

We also assume that there are two nonnegative numbers c and s, called the consumption and profit coefficients, respectively; they satisfy $0 \leq c \leq 1$ and $0 \leq s \leq 1$. We shall assume that the stockholders, firms, etc. consume a fraction c of their surplus and reinvest a fraction s. All investment are invested in industries which pay interest (or dividends) at the rate to be specified out of current surplusses.

As in [6] we let α and β be the expansion and interest factors, respectively, and let x be the $1 \times m$ intensity vectors and y the $n \times 1$ price vectors. These vectors are normalized to be probability vectors, as usual. Then we can state the equilibrium conditions that the economy-wide model is to satisfy as follows:

$$x\,[B - \alpha\,(A + c\,W)] \geq 0 \tag{1}$$

$$[B - \beta\,(A + s\,W)]\,y \leq 0 \tag{2}$$

$$x\,[B - \alpha\,(A + c\,W)]\,y = 0 \tag{3}$$

$$x\,[B - \beta\,(A + s\,W)]\,y = 0 \tag{4}$$

$$x\,B\,y > 0 \tag{5}$$

The first condition states that the output of a given time period must be enough to provide the inputs of the next time period plus the consumption of the workers; the second condition states that the value of the outputs must be enough to cover the capitalized value of the inputs plus the excess profits; condition (3) says that over-produced goods should obtain zero price, and condition (4) says that inefficient industries should be used with zero intensity. Finally, condition (5) says that the production-pricing structure must be such that something of value is produced.

We do not require any necessary relationship between c and s. Note that $c + s < 1$ corresponds to the case of hoarding, while $c + s > 1$ corresponds to the case of capital consumption. Either of these strategies may be pursued by economies at different stages of their development.

The model presented in this paper is capable of an alternative interpretation[3] appropriate for analyzing the effects upon the expansion factor and the interest rate of changes in the real wage rate and the propensity to consume. Let W^r be a row vector whose i-th component reveals the quantity of labor required to operate process i at unit intensity and W^c a column vector whose j-th component reveals the fraction of total consumption that workers allocate to good j. Further, let the scalar s be an index of real wages (with weights W^c) and c^* the fraction of income that workers consume rather than save. Then letting $W = W^c W^r$ and $c = c^* s$, the components of $c\,W$ in equation 1 reveal the consumption of good j by workers employed in process i when it is operated at unit intensity. Furthermore, the vector $s\,W\,y$ of equation 2 reveals the cost of labor,

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3. The authors are greatly indebted to Professor M. C. LOVELL for this interpretation.

industry by industry, incurred when each process is operated at unit intensity.

One can also envision 'mixed models' in which c, s, and W are divided between consumption and reinvestment of both stockholders and workers surplus income. We shall not specifically derive such models here.

As in our previous model [6], additional assumptions must be made in order that solutions to (1)–(5) must necessarily exist. These are:

$$v(B) > 0 \qquad (6)$$

$$v(-A) < 0 \qquad (7)$$

$$A + B + W > 0 \qquad (8)$$

The interpretations of (6) and (7) are as before, namely, (6) says that B has no zero columns which means that every good can be produced by some process, while (7) says that A has no zero rows meaning that every process requires a positive amount of some input. Condition (8) which is similar to the original von Neumann condition $A + B > 0$, prevents the economy from breaking up into disconnected sub-economies. Here it is an assumption of convenience and can easily be removed. But we make it in this paper since our principal interest is to discuss the economic implications of an economy that is either connected by technological requirements $(A + B > 0)$ or at least by ubiquitous tastes of the workers in all industries (condition (8)). In Section VI we shall show that the 'tastes' of the public sector may have a unifying effect on different areas of the economy.

To reiterate the above points, let us note that at least three kinds of 'sub-economies' may be distinguished: technological sub-economies; private consumption sub-economies; and public consumption sub-economies. Thus, looking at only one of these factors may permit breaking up the economy in various ways, but looking at them simultaneously may, in effect, tie them all together into a single economy. As far as we know, these ideas are stated here for the first time for economics in a technically precise manner. We would like to note the strong similarity of this procedure to the method of composition and decomposition of n-person games (cf. [13], as summarized on pp. 340–1 there).

III. ILLUSTRATION: AN ELEMENTARY EXAMPLE

In order to help the reader review the KMT model and also to illustrate the new model we present a simple three-industry, three-product economy first without and then with consumption and savings. In the model there are three products, Wheat, Entertainment, and Diamonds, and three industries, one that produces each of the products. The input and output matrices are:

	A matrix			B matrix		
	W	E	D	W	E	D
Wheat Industry	10	0	0	40	0	0
Entertainment Industry	2	10	0	0	20	0
Diamond Industry	1	0	10	0	0	12

In this model $\alpha = \beta$ and there are clearly three different feasible expansion (and interest) rates. These together with their solutions are:

$\alpha = 4; x = (1, 0, 0)$ and y is an arbitrary column probability vector

$\alpha = 2; x = (a, 1 - a, 0)$ where $1/6 \leq a \leq 1$, and $y = (0, b, 1 - b)$,

where b is an arbitrary number ≥ 0

$\alpha = 1.2; x = (a_1, a_2, 1 - a_1 - a_2)$ where $29.2\, a_1 + 3.6\, a_2 \geq 1.2$ and

$$a_1 \geq 0,\ a_2 \geq 0;\ \text{and}\ y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These solutions are somewhat unsatisfactory since when only the wheat industry is run, wheat has positive price, but if any other industry is also run, the price of wheat is zero.

Suppose now that the W matrix of the economy is given by

$$W = \begin{pmatrix} 0.6 & 0.4 & 0.1 \\ 0.8 & 0.2 & 0.2 \\ 0.9 & 0.3 & 0.3 \end{pmatrix}$$

and $c = 0.8$ and $s = 0.2$. We now see that the technological sub-economies in the previous example have vanished since $A + B + W > 0$ and there is only a single economy in which (8) holds. Thus, by per-

mitting the workers and stockholders in the various industries to take their remuneration partly in each of the three products of the economy, to consume part of it and invest part, we have tied together the economy and achieved a unique expansion and interest rate. We shall indicate the solution to this example in Section V and it will be seen that $\alpha < \beta$.

IV. EXISTENCE THEOREM

The purpose of this section is to show that the model defined by equations (1)–(5) always has a solution when assumptions (6)–(8) hold.

When $c = s$, equations (1)–(5) become those of the KMT model and the existence theorem given there implies the following result:

Lemma 1. If $c = s$ and assumptions (6)–(8) hold then there is a solution to (1)–(5) in which $\alpha = \beta$.

Eventually we shall see that we can obtain solutions to (1)–(5) for arbitrary s, c, and W satisfying the stated assumptions, provided (6)–(8) hold. For the present we shall draw some logical consequences of the assumption of existence of such solutions.

Lemma 2. Assume there is a solution to (1)–(5). It follows that,

$$\text{if } c \geq s \quad \text{then} \quad 0 < \alpha \leq \beta \tag{a}$$

$$\text{if } c \leq s \quad \text{then} \quad \alpha \geq \beta > 0 \tag{b}$$

Proof. These consequences follow immediately from the formulas:

$$\alpha = \frac{x B y}{x (A + c W) y} \tag{9}$$

$$\beta = \frac{x B y}{x (A + s W) y} \tag{10}$$

which are immediately deducible from (3) and (4). The positivity of α and β follows from (5), (6), and (7), as does the nonzero property of the denominators in (9) and (10).
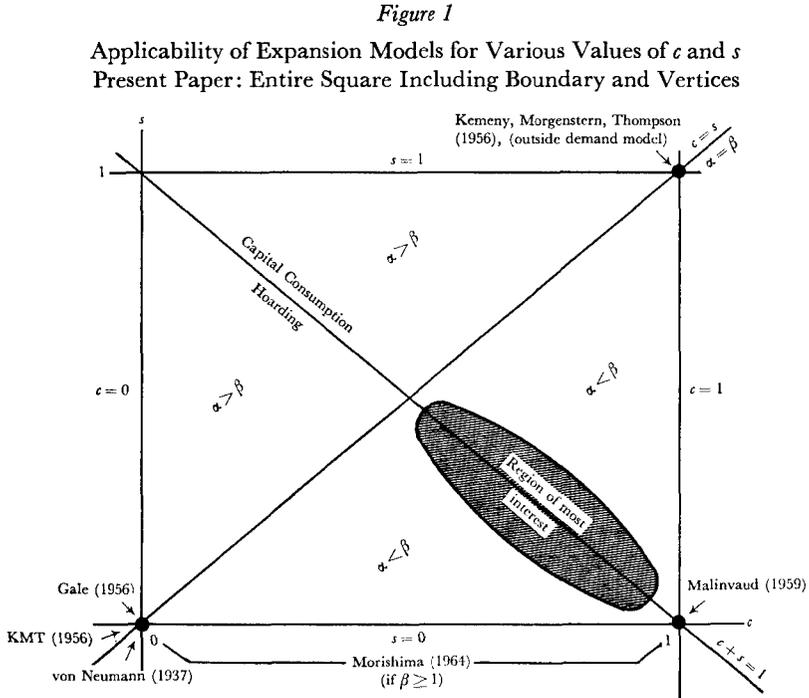
We now define the following matrix games:

$$M(\alpha) = B - \alpha (A + c W) \tag{11}$$

$$M(\beta) = B - \beta (A + s W) \tag{12}$$

and note that each of these matrices, considered by itself, is an ordinary expanding economy model which, by the above assumptions, will have a unique expansion factor which is equal to its interest factor. Let us denote these quantities by α' and β'.

Lemma 3. If there is a solution to (1)–(5) then $\alpha \leq \alpha'$ and $\beta \geq \beta'$. *Proof.* By the analysis of [6], the number α' is chosen so that the value of the game $M(\alpha')$ is zero. If (1) holds then the value of the game $M(\alpha)$ is positive or zero. Since A and W are nonnegative matrices and c is a nonnegative number, and since α' is unique, the inequality $\alpha > \alpha'$ would imply that the value of $M(\alpha)$ is negative contrary to the assumption that (1) holds. Hence, $\alpha \leq \alpha'$. The proof that $\beta \geq \beta'$ is similar.

The results of the first three lemmas permit us to draw *Figure 1* which illustrates the effects of c and s on the expansion and interest factors. The 45° line represents the situation in which $c = s$ and therefore $\alpha = \beta$. The point $c = s = 0$ represents the original von Neumann model [12], Gale's model [2] and the model with outside demand

Figure 1

Applicability of Expansion Models for Various Values of c and s
Present Paper: Entire Square Including Boundary and Vertices

that we discussed in [6]. The point $c = 1$, $s = 0$ represents the case discussed by Malinvaud. The line $s = 0$, $0 \leq c \leq 1$, is Morishima's model [9] on the assumption that $1 \leq \beta$. The other diagonal line of the square is the case for which $c + s = 1$, which is probably that of most economic interest. The additional reasonable assumption that $c > s$ indicates that the region in the lower right-hand triangle of the square is desired. Finally, if we restrict ourselves to the subset of this triangle that is close to the line $c + s = 1$, the area shaded in *Figure 1*, then we have the region of most probable economic interest.

Of course, the economist is much more interested in the corresponding variation of expansion and interest factors α and β. The corresponding figure is a mapping of the square in *Figure 1*, and in the course of the mapping some of the straight lines may become curves; see Example 4 and *Figure 2* in the next section. We do not know at present exactly how the general case is characterized, and this awaits further study and computation.

We now present three theorems that give various methods for finding solutions to (1)–(5). The first two theorems give sufficient conditions for the solutions to $M(\alpha)$ and $M(\beta)$ to be solutions to (1)–(5). The final theorem gives an algorithm for finding solutions even when the first two theorems do not apply.

Theorem 1. Let W be the constant matrix all of whose entries are the same number w; then the solutions x and y to $M(\alpha)$ and $M(\beta)$ are solutions to (1)–(5); the factors α and β are given by equations (9) and (10).

Proof. By the well-known theorem that adding the same constant to every entry of a matrix game does not change the optimal strategies the solutions to $M(\alpha)$ and $M(\beta)$ are identical and also solve (1)–(5) provided α and β are given by (9) and (10).

Theorem 2. Let x and α' be solutions to $M(\alpha)$ and y and β' be solutions to $M(\beta)$. Then if the positive components of x correspond to active[4] pure strategies in $M(\beta)$ and the positive components of y correspond to active pure strategies in $M(\alpha)$, these quantities $x, y,$ α', and β' are solutions to (1)–(5).

4. An active pure strategy is a pure strategy that is used with nonnegative weight in some optimal mixed strategy.

Proof. By assumption (1), (2), and (5) are satisfied. The remainder of the hypothesis proves that (3) and (4) are also satisfied.

> *Corollary.* If every strategy in $M(\alpha)$ and $M(\beta)$ is active then their solutions are also solutions to (1)–(5).

Example 5 in the next section shows that not every economy will satisfy one of these two conditions. Hence an algorithm for the general case is needed and is supplied in the next theorem.

> *Theorem 3.* Every model (1)–(5) satisfying (6)–(8) has an equilibrium solution.

Proof. We remark first of all that if $m = n = 1$ then the model clearly has a solution by either of the preceding two theorems.

Suppose m and n are > 1. The following algorithm will always suffice to find at least one, and possibly many, solutions.

1. First solve $M(\alpha)$ and $M(\beta)$ as ordinary von Neumann models. If the solutions satisfy either Theorems 1 or 2, halt. If not go to 2.

2. Add a constraint of the form $x_i = 0$ corresponding to a zero component of the solution for x in $M(\alpha)$; or else add a constraint of the form $y_j = 0$ corresponding to a zero component of the solution for y in $M(\beta)$. This is equivalent to crossing out a row of $M(\alpha)$ or else a column of $M(\beta)$. Go to 1.

The algorithm is certain to stop after a finite number of steps for it will never remove a single remaining row or a single remaining column since they never correspond to zero components. At each step either a row or a column is eliminated and this can clearly proceed only a finite number of times. At some point, either when there is exactly one row and one column left and the problem falls under the case above, or at some earlier state, the algorithm will terminate with a solution to the problem. See Example 5 in the next section for an application of the algorithm.

By working out an enumeration of all possible ways of adding the constraints in 2 one can determine all possible solutions to (1)–(5). Note that there will not necessarily be a unique solution, see again Example 5.

V. EXAMPLES

In order to show more clearly the complexity of the present model we give six examples that bring out various points of interest. The actual numbers chosen in these examples serve merely arithmetic convenience.

We first take three examples that use the same technological matrices and have various values of c and s. The technological matrices involved in each of these examples will be:

$$B = (5, 3), \quad A = (2, 1), \quad W = (2, 2)$$

We thus have one production process and two different goods. Our problem is, given c and s, to find an expansion rate, an interest rate, and a price structure that will satisfy (1)–(5). We shall solve this example for three different combinations of c and s.

Example 1. Suppose $c = 1/8$ and $s = 0$. This is a case of consumer hoarding. Then,

$$M(\alpha) = (5 - \frac{9}{4}\alpha, \ 3 - \frac{5}{4}\alpha)$$

$$M(\beta) = (5 - 2\beta, \ 3 - \beta)$$

From these we find that $\alpha' = 20/9$ and $\beta' = 5/2$. The column player's strategy in each game is $y = (1, 0)$. Hence, we can solve the model with $\alpha = \alpha' = 20/9$ and $\beta = \beta' = 5/2$, and with the first good having unit price and the second good free. Here $\alpha < \beta$.

Example 2. Suppose $c = 1$ and $s = 0$. Then,

$$M(\alpha) = (5 - 4\alpha, \ 3 - 3\alpha)$$

$$M(\beta) = (5 - 2\beta, \ 3 - \beta)$$

It is easy to show that $\alpha_c = 1$ with $y = (0, 1)$ and $\beta_s = 5/2$ with $y = (1, 0)$. These quantities will solve (1) and (2) but not (3) and (4). We must therefore change either α or β. It is impossible to decrease α without making (1) impossible to solve. Hence, we try to increase β

so that the price structure $y = (0, 1)$ will work. It is not hard to show that $\beta = 3$ will do this. Hence our solution is given by $\alpha = \alpha_c = 1$ and $\beta = 3 > \beta' = 5/2$ with $y = (0, 1)$. Again $\alpha < \beta$.

This is a classical example in which the workers consume all their earnings which results in an interest rate higher than the expansion rate.

Example 3. Suppose now that $c = 0$ and $s = 1$. This example is not economically realistic if W is interpreted as the wages of the workers since they cannot live on zero consumption. But it is an extreme example of the case in which their savings coefficient is greater than their consumption coefficient and is easy to work out mathematically. Then:

$$M(\alpha) = (5 - 2\,\alpha, 3 - \alpha), \qquad \alpha' = 5/2, \text{ and } y = (1, 0)$$

$$M(\beta) = (5 - 4\,\beta, 3 - 3\,\beta), \quad \beta' = 1, \quad \text{and } y = (0, 1)$$

Again equations (3) and (4) are not satisfied and we must choose different expansion or interest rates. As before, α cannot be changed, but increasing β to $5/4$ solves the situation. So our solution results with $\alpha = \alpha' = 5/2$, $\beta = 5/4 > \beta' = 1$ and $y = (1, 0)$. Here $\alpha > \beta$.

Example 4. Next we solve an example in which $\alpha = \alpha'$ and $\beta = \beta'$ always is a good choice of values. Let

$$B = \begin{pmatrix} 3 & 12 \\ 5 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Then we have

$$M(\alpha) = \begin{pmatrix} 3 - \alpha\,(1 + c) & 12 - \alpha c \\ 5 - \alpha\,(1 + c) & -\alpha\,(3 + 2\,c) \end{pmatrix}$$

and

$$M(\beta) = \begin{pmatrix} 3 - \beta\,(1 + s) & 12 - \beta s \\ 5 - \beta\,(1 + s) & -\beta\,(3 + 2\,s) \end{pmatrix}$$

If we choose

$$\alpha' = \frac{4}{1 + c}$$

then $x = (1/2, 1/2)$ is an optimal strategy in the resulting game

$$\begin{pmatrix} -1 & d \\ 1 & -d \end{pmatrix} \quad \text{where } d = \frac{12 + 8c}{1 + c}$$
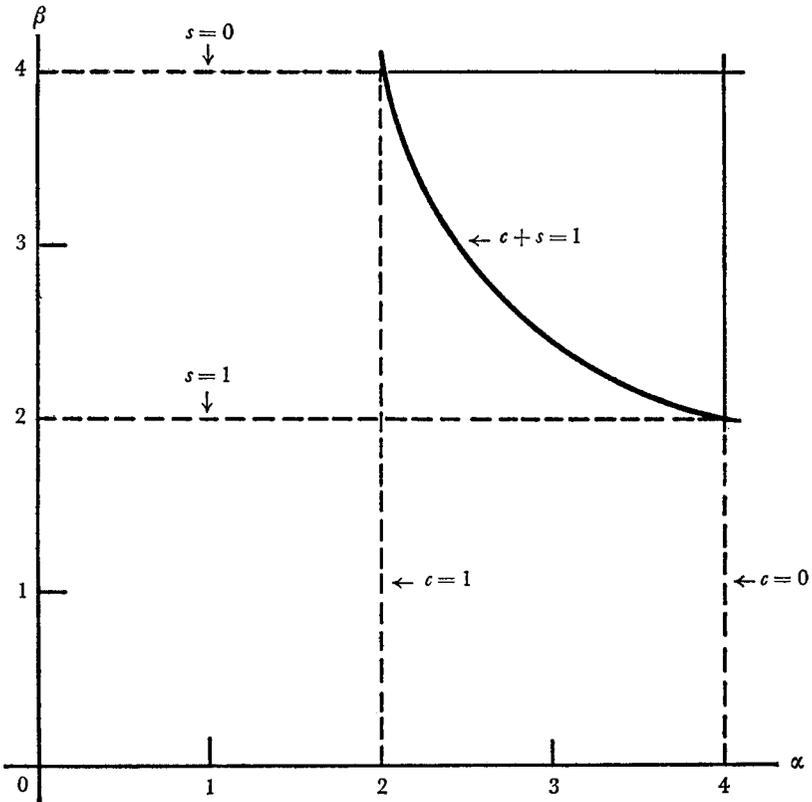
Similarly, if we choose

$$\beta' = \frac{4}{1 + s}$$

then $y = [e/(1 + e), \ 1/(1 + e)]$ is an optimal strategy in the resulting game

$$\begin{pmatrix} -1 & e \\ 1 & -e \end{pmatrix} \quad \text{where } e = \frac{12 + 8s}{1 + s}$$

Because both strategies are completely mixed, the other equations of the model are also satisfied for all c and s. In *Figure 2* we have sketched

Figure 2

the curve of β versus α for the condition $c + s = 1$. The curve has the equation

$$\beta = \frac{4\alpha}{3\alpha - 4}$$

and is therefore a hyperbola. Note that $c = 0$ and $s = 1$ (e.g., low consumption and high savings) results in $\alpha = 4$ and $\beta = 2$ (i.e., high expansion rate and low interest rate). Also, $s = 0$ and $c = 1$ are results in $\alpha = 2$ and $\beta = 4$ (e.g. low expansion rate and high interest rate). These results are intuitively reasonable.

Observe that the line $c + s = 1$ in *Figure 1* has been transformed into a hyperbola in *Figure 2*. No doubt other examples can be constructed to show more complicated changes.

Example 5. The next example demonstrates the necessity for the algorithm presented in the proof of Theorem 3. It shows that the solution to the model involves the choice of an optimal strategy for the row player in the game $M(\alpha)$, and an optimal strategy for the column player in the game $M(\beta)$. In the example, this pair of choices can be made in two different ways, each leading to a different economic solution of the model. Later, we shall remark further on the question of choice of solution.

The technological matrices for the example are:

$$B = \begin{pmatrix} 20 & 0 \\ 1 & 20 \end{pmatrix}, \quad A = \begin{pmatrix} 10 & 0 \\ 0.1 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 2 & 50 \end{pmatrix}$$

We choose $c = 0.1$ and $s = 0.9$ so that

$$M(\alpha) = \begin{pmatrix} 20 - 10.1\alpha & 0 \\ 1 - 0.3\alpha & 20 - 6\alpha \end{pmatrix}$$

$$M(\beta) = \begin{pmatrix} 20 - 10.9\beta & 0 \\ 1 - 1.9\beta & 20 - 46\beta \end{pmatrix}$$

The solution to the model $M(\alpha)$ is given by

$$\alpha = 10/3, \quad x = (0, 1), \quad \text{and} \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

while the solution to the $M(\beta)$ economy is

$$\beta = 20/10.9, \quad x = (1, 0), \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which are clearly incompatible. Utilizing the algorithm of the previous section we can impose either the constraint $x_1 = 0$ or $y_2 = 0$. The first constraint makes the above solution to $M(\alpha)$ be optimal with $\beta = 20/46$. The second constraint makes the solution to $M(\beta)$ be optimal with $\alpha = 200/101$. Hence we here have a case in which there are two and only two solutions which are quite different in character, and there is no reason to prefer one over the other.

The question of choice of solution arises. A choice is impossible within the rules of the game, i.e., there is no way of preferring one outcome over another. It would require an outside agency, say the government or some ideologically controlled board, to choose one rather than the other for reasons which have nothing whatsoever to do with the game as such. The choice is at any rate *not* made by members of the economy. Although this is a two-person set-up, the situation is reminiscent of the solution concept of a cooperative n-person game, where the solution consists by necessity of a set of more than one imputation which do not dominate each other and among which there is no choice possible. Game theory cannot indicate at present which of the imputations belonging to the solution set will actually materialize. It is noteworthy that a similar occurrence is to be observed here. This is fundamentally different from the processes described by classical mechanics and *a fortiori* by classical economics. (Cf. footnote 3, page 4.)

Applied to the present case the above result means that the frequently made assumption that the economy must automatically go to the efficiency point is not in general true for the simple reason that no such point necessarily exists. In the above counter-example there is no way of choosing between the two possible states unless the indicated outside factors are brought into play or entirely new ideas are formulated. That this example could be found, establishes a tie to the conceptual world of game theory when the latter is used as a model of economic reality rather than merely as a tool of mathematical analysis.

Example 6. We now solve the example given in Section III. The search algorithm of HAMBURGER, THOMPSON, and WEIL [4] was pro-

grammed for an electronic computer. This method solves the game $M(\alpha)$ by the following procedure: choose a number α' so that the value of $M(\alpha')$ is negative; choose another number α'' so that the value of $M(\alpha'')$ is positive; next find the value of $M(\alpha)$ where $\alpha = (\alpha' + \alpha'')/2$; if $v(M(\alpha)) > 0$, let $\alpha'' = \alpha$; if $v(M(\alpha)) < 0$, let $\alpha' = \alpha$; repeat; stop when a sufficiently accurate value of α has been determined.

The solution to the example was found using only a few seconds time of a moderately fast electronic computer. They are:

$$\alpha = 1.1705 \qquad \beta = 1.1927$$

$$Activity\ vector = (0.0689, 0.0343, 0.8968)$$

$$Price\ vector\ \ \ = (0.0009, 0.0061, 0.9930)$$

Thus we see that diamonds are priced very much more highly than either wheat or entertainment; entertainment is priced nearly seven times higher than wheat; the wheat industry is operated at nearly twice the level of the entertainment industry, but at less than a twelfth of the level of the diamond industry. Intensity of operation of an industry should, of course, be measured in terms of the level of sophistication of technique as well as physical effort.

The numerical data were chosen rather arbitrarily, but the above example suffices to show the following: (a) the demand matrix has tied together the economy so that unique expansion and interest rates occur, and (b) the model is capable of fast computation on an electronic computer. By the use of the fastest computers now available, it would be possible to find numerical solutions to models having values of m and n ranging in the hundreds.

VI. THE NEW MODEL: PUBLIC SECTOR

The reader will have observed that our model does not have money in it. (The question of how to introduce money into such models is, as far as we know, completely unsolved.) Our mechanism for paying workers and stockholders was to reimburse them in real goods, in proportions according to their tastes. Then we let them consume or

reinvest their holdings as they desired. The same may be done for the public sector; that is, we may permit the 'government' to acquire (say, by its tax power) part of the output of the economy in real goods according to its public 'preferences' and consume or reinvest it as it sees fit.

To be specific, let c' and s' be the consumption and savings coefficients of the public sector, and let G be the matrix which indicates the real goods payments of the economy to the government. We shall assume that c' and s' are nonnegative, but not necessarily bounded above by 1. It is well-known that governments can spend more than their total revenue. The equations that govern the new model are:

$$x\,[B - \alpha\,(A + c\,W + c'G)] \geq 0 \qquad (1')$$

$$[B - \beta\,(A + s\,W + s'G)]\,y \leq 0 \qquad (2')$$

$$x\,[B - \alpha\,(A + c\,W + c'G)]\,y = 0 \qquad (3')$$

$$x\,[B - \beta\,(A + s\,W + s'G)]\,y = 0 \qquad (4')$$

$$x\,B\,y > 0 \qquad (5')$$

We shall also make assumptions (6) and (7). However, we shall replace (8) by

$$A + B + W + G > 0 \qquad (8')$$

These eight conditions characterize the new model.

Although the new model differs in only relatively minor respects from the previous one, there are several important possibilities which it opens. We first note that condition (8') can hold even if (8) does not. In other words, the governmental 'tastes' can unify an economy that would otherwise be disconnected. In real economies it is observed that only the government is willing to pay for such things as roads, community welfare, defense weapons, etc. Second, the new model now permits the government, through its choice of c' to influence the expansion rate α, and through its choice of s' similarly to influence the interest rate β of the economy. Even further, the government can change G, by changing its tax policies, and influence both these factors simultaneously. Exactly how these changes come about is completely determined by the equations of the model. It is clear,

without formal proof, that increasing c' will decrease α, and increasing s' will decrease β, while multiplying the matrix G by a number larger than 1 will decrease both of these factors simultaneously. All of these conclusions seem to be in complete agreement with conclusions from other parts of economics.

We conclude by reworking the example of Section III assuming that the government has a taste matrix:

$$G = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$$

We also assume $c' = 1.2$ and $s' = 0$; i.e., a case of deficit spending. The solution to the model for the interest rate and price vector is exactly as in Example 6 of Section V, since $s' = 0$. The solution for expansion factor and activity vector, however, has been slightly changed to:

$$\alpha = 1.1660 \quad \text{and activity vector} = (0.0687, 0.0340, 0.8973)$$

Note that the activity vector is, to three decimal places of accuracy the same as before, while the expansion factor has been decreased by 0.005.

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